Theorem 1a, surface S_2 must be spherical. In the presence of friction at point A of contact of surfaces S_2 and S_3 the theorem's hypothesis (3) can obviously be satisfied only if the axis CZ' passes constantly through point A. Consequently, the projections of the velocities of the geometric points B and A onto plane BXY must, by hypothesis, be equal, which does not obtain in general. Theorem 3a is proved.

In conclusion we emphasize that the statements of Theorems 1a, 2a and 3a are valid when the relative velocities of the bodies at their points of contact equal zero; otherwise, the statements lose force. For example, under an appropriate choice of the moving axes the hypotheses of Theorem 3 are satisfied in the problem of the rolling of a body of arbitrary form over the absolutely smooth surface of a moving sphere [1].

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GENERALIZATION OF THE RAYLEIGH THEOREM TO GYROSCOPIC SYSTEMS

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V. F. ZHURAVLEV
(Moscow)

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The Rayleigh theorem on the properties of the spectrum of a linear conservative mechanical system is generalized to embrace the gyroscopic systems, i. e. to the case in which the equations of motion contain, in addition to the kinetic and potential energy matrices, an arbitrary skew-symmetric matrix of gyroscopic forces.

1. Linear gyroscopic system. We shall consider a linear gyroscopic system described by the following general expression:

$$Aq^{\cdot\cdot} + \Gamma q^{\cdot} + Cq = 0, \quad q \in R^n \tag{1.1}$$

where A is the kinetic energy matrix, C is the potential energy matrix, both A and C being symmetric $n \times n$ matrices, and Γ is a skew-symmetric matrix of the gyroscopic forces of the same dimension. The properties of the natural frequencies of the above system were investigated by Rayleigh in [1] for the case $\Gamma = 0$

Let $\Gamma \neq 0$. Following the Rayleigh case, we assume the kinetic and potential energies of the system to be positive definite forms with respect to velocities and coordinates: $(q^\cdot, Aq^\cdot) > 0$ for $q^\cdot \neq 0$, and $(q, Cq) > 0$ for $q \neq 0$. It can be shown that for any matrix Γ the characteristic equation of the system (1.1) has pure imaginary roots, consequently its particular solution can be written in the form

$$q = Ie^{i\lambda t} \tag{1.2}$$

Here λ is a real number the modulus of which is called the natural frequency of the system. It also represents one of the roots of the equation

$$\det(-A\lambda^2 + i\Gamma\lambda + C) = 0 \tag{1.3}$$

The above equation has $2n$ real roots, and it is evident that if λ_0 is a root of this equation, so is $-\lambda_0$. Thus the system has exactly n natural frequencies. The vector I corresponding to the particular root λ represents a nontrivial solution of the linear algebraic system

$$(-A\lambda^2 + i\Gamma\lambda + C)I = 0 \tag{1.4}$$

The complex solution of this system can be written in the form $I = p + ir$. Separating in (1.4) the real and imaginary parts, we obtain

$$(T\lambda^2 + G\lambda - V)x = 0, \quad x = \operatorname{col}\{p_1, \ldots, p_n, r_1, \ldots, r_n\} \in R^{2n} \tag{1.5}$$

$$T = \begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix}, \quad G = \begin{vmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{vmatrix}, \quad V = \begin{vmatrix} C & 0 \\ 0 & C \end{vmatrix}$$

and here all three matrices T, G and V are symmetric.

Besides the mechanical system (1.1), we can also consider a mechanical system of twice the dimension which is no longer gyroscopic, and the potential energy matrix of which is negative definite

$$\frac{d}{dt}\frac{\partial(S^\cdot, TS^\cdot)}{\partial S^\cdot} + \frac{\partial(S^\cdot, GS^\cdot)}{\partial S^\cdot} - \frac{\partial(S, VS)}{\partial S} = 0 \tag{1.6}$$

Seeking its solution in the form $xe^{\lambda t}$, we again arrive at Eqs.(1.5).

Some of the properties of the solutions of (1.5) are established by

Lemma 1. Let λ_0 be a certain root of Eq.(1.3), and x_+ a solution of system (1.5) corresponding to this root. Then a solution x_- of the system (1.5) exists for $\lambda = -\lambda_0$ such, that the following relations hold:

1) $(x_+, Tx_+) = (x_-, Tx_-), \quad (x_+, Vx_+) = (x_-, Vx_-)$
2) $(x_+, Gx_+) + (x_-, Gx_-) = 0$

Proof. We shall show that the vector

$$x_- = \begin{vmatrix} E & 0 \\ 0 & -E \end{vmatrix} x_+$$

where E is a unit matrix, satisfies all the above conditions. Let us substitute this expression into (1.5) in which we set $\lambda = \lambda_0$, and left multiply the result by the matrix E^*. This yields

$$(E^*TE^*\lambda_0^2 - E^*GE^*\lambda_0 - V)x_+ = 0, \quad E^* = \begin{vmatrix} E & 0 \\ 0 & -E \end{vmatrix}$$

It is easy to see that

$$E^*TE^* = T, \quad E^*VE^* = V, \quad E^*GE^* = -G \tag{1.7}$$

This yields the equation for the vector x_+, which turns out to be an identity. Thus the expression for x_- given above is in fact a solution of (1.5) for $\lambda = -\lambda_0$. Let us check the property (1). By (1.7) we have $(x_-, Tx_-) = (E^*x_+, TE^*x_+) = (x_+, E^*TE^*x_+) = (x_+, Tx_+)$; the property (2) is checked in a similar manner.

2. Characteristic function. Let us scalar multiply (1.5) by x

$$(x, Tx)\lambda^2 + (x, Gx)\lambda - (x, Vx) = 0 \tag{2.1}$$

Transferring the term containing $(x, Gx)\lambda$ to the right-hand side and squaring both sides, we obtain

$$(x, Tx)^2R^2 - [2(x, Tx)(x, Vx) - (x, Gx)^2]R + (x, Vx)^2 = 0, \quad R = \lambda^2 \tag{2.2}$$

If $x = x_+$ is a solution of (1.5) for $\lambda = \lambda_0$, then solving the quadratic equation (2.2) we obtain an expression for the square of the root, namely $R = \lambda_0^2$. If, on the other hand, x in (2.2) is an arbitrary vector, then the relation defines the implicit function $R(x)$. We shall call this function the characteristic function of the system (1.1), or of the system (1.6).

Lemma 2. The solutions of the algebraic system, and only these solutions, represent the critical points of the function $R(x)$.

Proof. We differentiate the relation (2.1) with respect to x, assuming that $\lambda = \lambda(x)$

$$\lambda^2Tx + \lambda Gx - Vx + \left[(x, Tx)\lambda + \frac{1}{2}(x, Gx)\right]\frac{d\lambda}{dx} = 0 \tag{2.3}$$

We shall show that the expression appearing in (2.3) within the square brackets is not zero when $x \neq 0$. Let us assume the opposite. Then a value $x = x_0 \neq 0$ can be found such, that

$$\lambda(x_0) = -{}^1/_2\,(x_0, Gx_0) / (x_0, Tx_0)$$

Substituting this value of $\lambda(x_0)$ into (2.1), we obtain

$$(x_0, Gx_0)^2 + 4(x_0, Tx_0)(x_0, Vx_0) = 0$$

which is impossible since T and V are both positive definite. Consequently $d\lambda / dx$ vanishes only for those x which are solutions of the system (1.5). On the other hand, from (2.1) we find that, by virtue of their positive definiteness T, V and $\lambda(x) \neq 0$, if $x \neq 0$. Therefore $dR / dx = 2\lambda d\lambda / dx$ becomes zero whenever $d\lambda / dx$ does.

Lemma 3. Function $R(x)$ (every one of its branches) is a monotonously increasing function of the potential energy, i.e. if we have two mechanical systems with the same T and G and with V^* and V such that $(x, Vx) < (x, V^*x)$ for any value of x, then $R(x) < R^*(x)$.

Proof. We introduce the notation $(x, Tx) = t$, $(x, Vx) = v$, $(x, Gx) = g$. From (2.2) we obtain

$$R = \frac{2tv + g^2 \pm g\sqrt{g^2 + 4tv}}{2t^2} \tag{2.4}$$

which on differentiating yields $dR / dv > 0$ for $x \neq 0$, as $4tv > 0$.

3. Characteristic surface. We shall assume that the following metric is defined in R^{2n} in terms of the quadratic form (x, Tx)

$$\|x\| = \sqrt{(x, Tx)}$$

We consider, in this space, a hypersurface Π defined by the equation

$$\| x \|^4 R(\dot{x}) = 1 \tag{3.1}$$

Substituting $R(x)$ from the above equation into (2.2), we obtain the following equation for this surface:

$$(x, Gx)^2 - (x, Vx)^2 (x, Tx)^2 + 2(x, Tx)(x, Vx) = 1 \tag{3.2}$$

Equation (3.2) determines a closed surface of the eighth order. A ray projected from the coordinate origin in any direction, intersects this surface at two points corresponding to the different signs in (2.4).

Lemma 4. Vector x_0 satisfying the system (1.5) of length $|\lambda_0|^{-1/2}$, where λ_0 is the eigenvalue corresponding to this vector, belongs to Π.

Proof. Substituting $\|x_0\|$ into (3.1) and remembering that $R(x_0) = \lambda_0^2$ for one of the branches, we obtain an identity.

We shall call the length of this vector the principal semiaxis of the surface.

In what follows, we shall distinguish between the two branches of the surface Π, denoting the branch corresponding to the plus or minus sign in (2.4) by Π_+ and Π_-, respectively. We shall use the same notation for the branches of $R(x)$, namely $R_+(x)$ and $R_-(x)$.

Lemma 5. The surfaces Π_+ and Π_- have the same, common system of principal semiaxes.

Proof. From (2.4) and Lemma 1 we obtain $R_+(x_+) = R_-(x_-)$, $R_+(x_-) = R_-(x_+)$, and this implies that the linearly independent system of solutions (1.5) consisting of $2n$ vectors, can be divided into two subsystems

$$x_1', \ldots, x_n', \quad x_1'', \ldots, x_n'' \tag{3.3}$$

such that $R_+(x_i') = \lambda_i^2$ and $R_-(x_i'') = \lambda_i^2$, which completes the proof.

We shall assume that the eigenvalues corresponding to the system of vectors (3.3) introduced in Lemma 5, are distributed in the order of their increasing moduli: $|\lambda_1| \leqslant \ldots \leqslant |\lambda_n|$. Let us denote the principal semiaxes by

$$a_i = \| x_i' \| = \| x_i'' \| = |\lambda_i|^{-1/2}$$

Clearly we have

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \tag{3.4}$$

Since each of the branches Π_+ and Π_- has the same system of semiaxes (3.4), it will be sufficient to consider only one of these branches, e.g. Π_+. We shall show that the principal semiaxes of Π_+ have extremal properties. Let $R^{2(n-m+1)}$ be a subspace stretched over the vectors $x_m', x_m'', \ldots, x_n', x_n''$ the norm of which is equal to or less than a_m.

Lemma 6. $a_m = \max x$ for $x \in \Pi_+ \cap R^{2(n-m+1)}$.

Proof. Consider the mechanical system (1.6) with additional constraint $S \in R^{2(n-m+1)}$. The system is equivalent to a mechanical system with $2(n - m + 1)$ degrees of freedom for which the forms $(S^\cdot, TS^\cdot)$, $(S^\cdot, GS^\cdot)$ and (S, VS) are equal to the bounds of the corresponding forms of the system without a constraint on $R^{2(n-m+1)}$. Consequently the bound of the eigenfunction $R(x)$ on $R^{2(n-m+1)}$ will be equal to the eigenfunction of the system with the constraint. Such an eigenfunction will have exactly

$2(n-m+1)$ critical points. On the other hand, the vectors forming the linear envelope of $R^{2(n-m+1)}$ will obviously be the eigenvectors for the bound of the eigenfunction on $R^{2(n-m+1)}$ and, since there are $2(n-m+1)$ of these vectors, they and only they will satisfy the necessary conditions for the extremum of the function $R(x)$ for $x \in R^{2(n-m+1)}$. Moreover λ_m^2 is the absolute minimum of the function $R(x)$ on $R^{2(n-m+1)}$, and from this it follows by virtue of (3.1) that a_m is the absolute maximum of the norm of the radius vector of the surface Π_+ on $R^{2(n-m+1)}$.

Let us consider the intersection of the surface Π_+ by some subspace R^{2m}, and introduce the notation $b_m = \min \| x \|$ for $x \in R^{2m} \cap \Pi_+$.

Lemma 7. For any R^{2m}: $b_m \leqslant a_m'$ and $\max_{R^{2m}} b_m = a_m$.

Proof. Let $R^{2(n-m+1)}$ be the subspace appearing in Lemma 6. Since the sum of the dimensions of R^{2m} and $R^{2(n-m+1)}$ is greater than $2n$, the above subspaces intersect. Let $x \in R^{2m} \cap R^{2(n-m+1)} \cap \Pi_+$, then by Lemma 6 $\| x \| \leqslant a_m$; on the other hand $\| x \| \geqslant b_m$ since $x \in R^{2m}$, and this implies that $b_m \leqslant a_m$. It is obvious that $\max_{R^{2m}} b_m = a_m$ since the upper edge is attained on the subspace which is a linear envelope of the vectors $x_1', x_1'', \ldots, x_m', x_m''$.

4. Theorem on the behavior of the eigenfrequencies under varying rigidity. Let two mechanical systems of the form (1.1) be given with the same matrices of the kinetic energy and gyroscopic forces, with the potential energy matrices denoted by C and C^*. We shall call a system more rigid if its potential energy is greater: $(q, C^*q) > (q, Cq)$ for any $q \neq 0$.

Theorem. When the rigidity of the system (1.1) in which A and C are positive definite and Γ is an arbitrary skew symmetric matrix, is increased, then all natural frequencies of the system can only increase.

Proof. From the inequality $(q, C^*q) > (q, Cq)$ follows the inequality $(x, V^*x) > (x, Vx)$, and from this we have, by virtue of Lemma 3, $R_+(x) < R_+^*(x)$ for all $x \neq 0$ By virtue of (3.1), this means that Π_+^* lies completely within Π_+. Let us consider the intersection of the surfaces Π_+ and Π_+^* by the subspace R^{2m}. Clearly $b_m \geqslant b_m^*$, consequently $\max b_m \geqslant \max b_m^*$. But $\max b_m = a_m$, and $\max b_m^* = a_m^*$, therefore $a_m \geqslant a_m^*$. Using the relation $a_m = |\lambda_m|^{-1/2}$, we find that $|\lambda_m| \leqslant |\lambda_m^*|$ for any m.

5. Corollaries.

Corollary 1. When $\Gamma = 0$, the above theorem becomes the Rayleigh theorem.

Corollary 2. We shall call a system the large-mass system if for any $q^{\cdot} \neq 0$ its kinetic energy is larger: $(q^{\cdot}, A^*q^{\cdot}) > (q^{\cdot}, Aq^{\cdot})$, and we have the following theorem: when the mass of the system increases, all its natural frequencies can only decrease.

To prove this theorem it is sufficient to show, similarly to Lemma 3, that $R(x)$ decreases monotonously for any x, with increasing kinetic energy.

Corollary 3. The requirement of the positive definiteness of C can be replaced by the requirement of nonnegativeness: $(q, Cq) \geqslant 0$ for any q.

In fact, the theorem holds for the case when some of the eigenvalues of the matrix C are arbitrarily small. By virtue of the continuous dependence of the roots of the characteristic equation on its coefficients, the theorem holds also in the limit, i.e. in the case when some of the eigenvalues of the matrix C are zero.

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ON THE EXISTENCE CONDITIONS FOR THE PARTICULAR JACOBI INTEGRAL

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I. A. KEIS
(Tallin)

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For certain nonholonomic and holonomic mechanical systems we have obtained the existence conditions for the particular integral being a linear bundle of the Hamiltonian and the momenta. The conditions are simplified for a certain class of holonomic systems, containing the well-known [1] case of the particular Jacobi integral. An example of the fulfillment of these conditions is a variant of the restricted problem of translation-rotational motion of a gyrostat in a Newtonian force field.

1. We consider a mechanical system S with the Lagrangian $L = T + U + N$. The linear function N of generalized velocities is the Meyer potential [2—4] of certain electromagnetic and gyroscopic forces. We separate the system S with the position coordinate vector $\mathbf{y} = (x_i, z_r)^*$ into subsystems S' and S'' with vectors $\mathbf{x} = (x_i)^*$ and $\mathbf{z} = (z_r)^*$

$$i = 1, 2, \ldots, l; \; r = 1, 2, \ldots, p; \; 1 \leqslant l, \; 1 \leqslant p; \; \dim y = l + p = n$$

We write the Lagrangian of system S as the sum

$$L = L_2' + L_1' + L_2'' + L_1'' + L^* + L_0 \tag{1.1}$$
$$L_2' = {}^1/_2 l_{ij}'(t, \mathbf{y}) x_i^\cdot x_j^\cdot, \quad l_{ij}' = l_{ji}', \quad \| l_{ij}' \| > 0 \quad (i, j = 1, 2, \ldots, l)$$
$$L_1' = l_j'(t, \mathbf{y}) x_j^\cdot, \; L_2'' = {}^1/_2 l_{rs}''(t, \mathbf{y}) z_r^\cdot z_s^\cdot, \; l_{rs}'' = l_{sr}'' \; (r, s = 1, 2, \ldots, p$$
$$L_1'' = l_r(t, \mathbf{y}) z_r^\cdot, \quad L^* = l_{ir}(t, \mathbf{y}) x_i^\cdot z_r^\cdot, \quad L_0 = L_0(t, \mathbf{y}), \quad f^\cdot = df / dt$$

Here and below summation is carried out over like indices and the superscript zero signifies the result of a substitution

$$f^\circ = f^\circ(t, \mathbf{x}, \mathbf{x}^\cdot) = f(t, \mathbf{x}, \mathbf{x}^\cdot, \mathbf{r}(t), \mathbf{v}(t))$$
$$(\mathbf{z} = \mathbf{r}(t), \mathbf{z}^\cdot = d\mathbf{r} / dt = \mathbf{v}(t))$$

We denote $\mathbf{r}(t)$, $\mathbf{v}(t)$ as the known motion of subsystem S'', for which the cylinder $\mathbf{z} = \mathbf{r}(t)$, $\mathbf{z}^\cdot = \mathbf{v}(t)$ is an invariant set of motions of S. In particular, the motion $\mathbf{r}_*$, $\mathbf{r}_*^\cdot$ of subsystem S'' possesses this property if system S has the particular invariants h_s $(s, r = 1,2, \ldots p)$

$$h_s(t, \mathbf{z}) = 0, \quad \det \| \partial h_s \mid \partial z_r \| \neq 0 \quad (h_s(t, \mathbf{r}_*(t)) \equiv 0)$$

or if the motions of S' have no effect on S''.